REyF Revista de Economía y Finanzas

www.reveyf.es

2024 – Vol. 2 – n.º 4 - Artículo 4

Endogenous Aggregate Beliefs: Equity Trading under Heterogeneity in Ambiguity Aversion

Irasema Alonso[a] **and Mauricio Prado**[b]

[a] *Columbia University. New York. USA.*

[b] *Copenhagen Business School. Copenhagen. Denmark.*

JEL CODES:
E0; G11; G12; D81

KEYWORDS:
Ambiguity aversion; Agent heterogeneity; Asset pricing; Portfolio choice; Wealth distribution

Abstract: We build a simple and tractable model of consumer heterogeneity in ambiguity aversion and use it to illustrate how asymmetric beliefs in the asset markets affect the dynamics of asset pricing, portfolio allocation, and the wealth distribution. The model is an otherwise canonical exchange-economy setting with two aggregate states of nature and two assets. The key focus is on how asset prices, when subjected to ambiguity, behave very differently in the short run than in the long run. In the short run, heterogeneity plays an important role. In the long run, belief heterogeneity persists, but the wealth distribution has evolved so that only the least ambiguity-averse investors matter for prices. As part of the short-run dynamics, the model can generate endogenous aggregate movements in non-participation—a drastic form of trading less—in response to ambiguity, with strong depressing effects on asset prices.

CÓDIGOS JEL:
E0; G11; G12; D81

PALABRAS CLAVE:
Aversión a la ambigüedad; Heterogeneidad de los agentes económicos; Valoración de los activos financieros; Elección de la cartera de inversión; Distribución de la riqueza

Resumen: Construimos un modelo simple y manejable con heterogeneidad en la aversión a la ambigüedad que presentan los consumidores y lo usamos para ilustrar como unas creencias asimétricas en los mercados de activos financieros afectan la dinámica de los precios de los activos, la asignación de la cartera de inversión, y la distribución de la riqueza. Por lo demás, la configuración del modelo es una economía canónica de intercambio con dos estados de la naturaleza agregados y dos activos. El enfoque clave es como los precios de los activos, cuando están sujetos a un entorno con ambigüedad, se comportan de una manera muy diferente en el corto plazo relativo al largo plazo. En el corto plazo, la heterogeneidad juega un papel importante. En el largo plazo, la heterogeneidad en la creencia persiste, pero la distribución de la riqueza ha evolucionado de tal manera que solo los inversores con la menor aversión a la ambigüedad pueden influir los precios de los activos. Como parte de la dinámica a corto plazo, el modelo puede generar movimientos endógenos agregados de no participación—una forma drástica de invertir menos—como respuesta a la ambigüedad, con fuertes efectos depresivos a los precios de los activos.

We are grateful to Per Krusell for very insightful comments. We also thank Hal Cole, David DeJong, and participants in seminars and conferences at IIES, University of Pittsburg, the Society of Economic Dynamics Meeting, and the North American Summer Meeting of the Econometric Society. Prado gratefully acknowledges financial support from Jan Wallander's and Tom Hedelius' Research Foundation. Any errors are, of course, ours.

E-mail: Ia2283@columbia.edu; mp.eco@cbs.dk
https://doi.org/10.32826/reyf.v2i4.365

1. Introduction

The financial crisis of 2008 has redirected attention toward the functioning of financial markets. From a perspective of understanding how financial markets work, then, what can be said from a theoretical perspective about possible market failures: their sources and aggregate effects? Answers to these two broad questions are of course important not just from a normative point of view—as guidance for regulatory analysis—but also from a positive one. In the rather large literature that has emerged in response to the need to generate answers, a common theme is that financial markets are fraught with informational asymmetries. The present paper is a contribution in this vein, not directly attempting to explain a recession event but rather to formulate a very simple and tractable model that allows us to think about asset trading under asymmetric information and follow the dynamics over a transition period characterized by significant ambiguity among parts of the traders. By design, in the long run of our economy, asset pricing is "normal" in that the ambiguity-averse traders will be negligible in terms of their relative wealth, but in the short run they can affect prices greatly.

More precisely, the aim here is to construct an illustrative model of asset trade that combines some important elements of the literature (heterogeneity and ambiguity) in one. Quite centrally, agents differ in their degree of ambiguity aversion. Ambiguity aversion is an assumption on primitives—on consumer/investor preferences—that generates heterogeneity in "ex post beliefs". So, we consider a very simple endowment model and assume that a fraction of the agents displays a given amount of ambiguity aversion while the rest (the "standard agents") do not. In order to make the analysis as simple as possible, we specialize to a logarithmic period utility function and *iid* and symmetric shocks. This setting allows us to think about asset pricing that can be highly influenced by heterogeneity, with implications that are very different in the short and long run. As in Sandroni (2000), the long-run relative wealth of the ambiguity-averse agents approaches zero, so the interesting asset-pricing implications take place during the transition.

For the particular case we look at, the heterogeneity and general-equilibrium effects from trading imply that the ambiguity-averse agents buy less of the risky asset in the short run—and instead invest in the risk-less asset. Over time, this tendency is increased, and the ambiguity-averse group becomes non-participants in the stock market, again as a result of trading with the other agents. The reason why this effect becomes stronger over time is that the wealth distribution is affected by trading, systematically making the ambiguity-averse agents a smaller and smaller proportion of the market in terms of wealth, thus influencing prices less and less. The ambiguity-averse agents lose wealth in relative terms because they are reluctant to buy risky assets and thus earn a return that is lower than the average return in the whole market. The fact that agents' influence on prices goes through their relative wealth position comes in large part from that assumption that agents have isoelastic utility functions, where wealth effects are important. Thus, the non-participation on the part of some agents in the economy arises endogenously through assumptions on preferences influencing portfolio choice in conjunction with the general-equilibrium effects arising from heterogeneity.

Our paper is, in some sense, also a comment on earlier results (by ourselves and others) showing that in economies where agents are ambiguity-averse, asset prices can have features more in line with data—that generate a higher equity premium/a lower risk-free rate—than the standard model where agents are averse to risk but do not feature ambiguity aversion. The comment, then, is that with heterogeneity, market trading between the heterogeneous agents will slowly tilt prices in the direction of the "standard" agents, thus gradually making the effect of ambiguity aversion smaller and smaller. Hence, the effects of ambiguity on asset prices are mainly present in the short run, or to the extent that the economy just experienced an increase in ambiguity.[1] Ambiguity aversion shows great potential in providing new asset-pricing implications and in allowing us to think of a reason why the elimination of aggregate fluctuations might be quite valuable but since, in our stationary model, the relative wealth of ambiguity-averse agents goes to zero in the long run, heterogeneity in the degree of ambiguity aversion puts some limits to these mechanisms.

The structure of the paper is as follows. In Section 2 we give some motivation behind the focus on asymmetric information and on ambiguity aversion and we also make some brief comments on how the present paper relates to the literature. Section 3 describes the economic environment. Section 4 contains our main analysis of the case with heterogeneous agents. Section 5 concludes, and the Appendix describes asset pricing in a representative economy with both *iid* and serially correlated shocks.

2. Motivating background and relation to the literature

Many observers suggest that the "Great Recession" of 2007-2010, whose background and main channels of operation took place in the financial sector of the economy, broadly defined—should lead us to reject modern macro theorizing (perhaps in favor of traditional Keynesian analysis). A different reaction is to (i) not reject the basic notion of models that rely on explicit microfoundations, i.e., using explicit decision theory and market descriptions; and (ii) instead move the research focus to models which potentially can deliver the kinds of financial-market outcomes that we have observed. Much ongoing research, based on different hypotheses, indeed take this route; a typical approach is to explore settings with asymmetric information. Informational frictions are a natural starting point for models of asset trade; information asymmetries indeed have often been described as central in the unfolding of the crisis. The purpose of the present paper is to further explore one avenue that allows us to look at heterogeneity in "induced beliefs": we aim to study the potential importance of

[1] See Epstein and Schneider (2007), for a model where ambiguity is time-varying.

ambiguity aversion and, in particular, heterogeneity among consumers/traders in ambiguity aversion.

Ambiguity aversion, and differences therein, is potentially a very useful, and tractable, way of modeling information asymmetries. The reason is that consumers do not need to possess different information: it is sufficient that they are "unsure" about probabilities and that they therefore, due to heterogeneity, end up acting according to different beliefs. That is, even though consumers are rational (but ambiguity-averse) and have the same information, they (rationally) choose different portfolios and, more generally, act differently vis-à-vis risk. In this sense, their beliefs are different in their "reduced form". Using ambiguity instead of literally using asymmetric information has the advantage that one does not need to model the extent to which (and how, exactly) markets reveal some of the information asymmetries, a challenge that is always present in such models and that limits tractability significantly. This is not to say that there is no tractable way of studying asymmetric information—the literature certainly contains several interesting analyses of this sort—but rather that studies based on ambiguity aversion and heterogeneity are a complementary route and one that does offer some potential advantages (but surely costs as well).[2]

More generally, what is the role for heterogeneity in consumer tastes for understanding macroeconomic phenomena? Indeed, individual decision-making is a key building block in micro-founded macroeconomic theorizing, though most of this theorizing relies on the "representative-agent" construct. Thus, the analysis relies on the premise either that, for the question at hand, an aggregation theorem holds, or that any quantitative departure from it does not significantly alter the conclusions of the analysis. Some abstractions from heterogeneity are surely easy to motivate, but others, we think, do deserve scrutiny, especially when the taste heterogeneity impacts importantly on the decisions macroeconomists build their models around. In this paper the focus is on understanding asset prices and wealth inequality. If different portfolios deliver different return characteristics, and if these characteristics have sufficiently different average returns, then in scenarios when different consumers have different portfolios of assets—a presumption that appears very much in line with data—one might well obtain sharp implications both for asset prices and for the long-run distribution of wealth. In this paper, we consider one possible determinant of different portfolio preferences, modeled through taste differences, and we look at its long-run effects on asset prices and wealth inequality: differences in the degree of *ambiguity aversion*.

Ambiguity aversion is a way of thinking more generally about "belief formation", and it has been used recently for understanding how risky assets might also be perceived as ambiguous and thus command lower prices and higher returns. Heterogeneity in ambiguity aversion is therefore a way of thinking about the possibility that consumers, despite their being rational, might have different induced beliefs about asset returns, thus leading to differences in their chosen portfolios. Since the ambiguity aversion we consider here is firmly based on axioms (see below), we are thus studying how differences in tastes might help us understand aggregates. Other approaches to understanding the observed heterogeneity in consumers' portfolios include (i) heterogeneous needs for risk sharing, (ii) heterogeneity in consumers' costs of accessing different financial instruments, and (iii) heterogeneity in information. Surely heterogeneity in risk sharing needs and transactions costs are important; the present analysis should be viewed as a complement in that regard. As for differences in information across people, they are likely very important too, and they are closely related to the object of study here, since they too lead to differences in consumer beliefs. However, they tend to be quite difficult to study in general equilibrium, since once needs to provide an analysis of why the price system does not reveal all privately held information. When it comes to the long-run effects on wealth inequality, in particular, we know of no study of it that is based on market interaction with partially revealing prices. Thus, one can perhaps view our current work as an alternative, and more tractable, way of dealing with belief differences in general equilibrium. Moreover, unlike in most representative-agent studies of ambiguity aversion, where equilibrium beliefs end up being determined very directly based on the modeler's choice of the typical parameter for ambiguity aversion ("a" below), in a model with heterogeneity in ambiguity aversion beliefs evolve endogenously and entirely nontrivially as a function of the market interaction, as we show in this paper.

Ambiguity aversion is a way of formalizing preferences that are consistent with the Ellsberg paradox, and it captures a form of violation of Savage's axioms of subjective probability. Instead, consumers behave as if a range of probability distributions are possible and as if they are averse toward the "unknown". With the typical parameterized representation of ambiguity aversion, consumers have maxmin preferences, thus maximizing utility based on the worst possible belief within some given set of feasible beliefs. In an economy with a small amount of randomness, there are first-order effects on utility if there is ambiguity about this randomness. Thus, ambiguity aversion is in contrast to the standard model, where risk aversion leads to second-order effects on utility.

Previous papers, such as our own work, have analyzed the role of ambiguity aversion for asset prices, but in representative-agent settings. Alonso and Prado (2015) looks at asset pricing in a simple Mehra-Prescott-style endowment economy.[3] There, we demonstrate how larger equity premia can be obtained by assuming ambiguity aversion, along with low risk-free rates. The key parameter in the model is the amount of ambiguity aversion, but it interacts nonlinearly with other parameters, such as the coefficient of relative risk aversion. We restrict preference parameters

[2] Interesting and tractable ways of studying informational asymmetries include Angeletos and La'O (2009), Hatchondo (2008), Hatchondo, Krusell, and Schneider (2014), Fostel and Geanakoplos (2008).

[3] Epstein and Miao (2003) have a model in the same spirit as ours, but they identify agents as countries, and the model is applied to address the consumption home bias and equity home-bias puzzles; a similar paper is Alonso and Sysuyev (2009). Thus, although the frameworks in those other papers are similar to that here, their focus is different than ours.

in order to as well as possible match some asset-price facts – the equity premium and a short-term risk-free return – and then compute the welfare benefits of removing all consumption fluctuations given those parameters[4]. A question we thus ask with the present paper is to what extent the results in this previous work concerning asset prices are robust to introducing heterogeneity in ambiguity aversion.

As for the survival of heterogeneity over time, in our paper, we assume a simple time-separable preference and show that ambiguity-averse agents disappear economically in the very long run. In contrast, Borovička (2015) has shown that when using non-separable preferences, e.g., homothetic recursive preferences of the Duffie-Epstein-Zin type, ambiguity-averse agents may survive economically even in the long run. Condie (2008) shows that without aggregate risk, ambiguity-averse investors may survive in the long run if they always believe that the true distribution could be wrong in many possible directions. Condie also proves that if an economy exhibits aggregate risk and there exists an expected-utility investor with correct beliefs, then any investor with recursive multiple-prior utility who satisfies the strict minimum consensus property will almost surely vanish. In our model, we have aggregate risk, and the ambiguity-averse agent is always "pessimistic" under our specification. Guerdjikova and Sciubba (2015) show that consumers with decreasing absolute ambiguity aversion whose prudence with respect to ambiguity exceeds twice their absolute ambiguity aversion almost surely survive in the presence of expected utility maximizers with correct beliefs. If the economy further exhibits aggregate risk, they drive the expected utility maximizers out of the market and determine prices in the limit. Colacito and Croce (2014) look at a model with two goods and a specific recursive Epstein-Zin preference representation with early resolution of uncertainty and provide conditions under which the solution of the planner's problem exists and features a non-degenerate invariant distribution of Pareto weights. Thus, these results are also different than those we obtain—through the assumptions just mentioned. Of course, the focus here is different in other ways; our emphasis is more on the dynamic endogeneity of beliefs and portfolios. But overall, it is clear that various combinations of assumptions lead agents to either lose their impact on prices/lose their wealth or not. Our particular modeling assumptions lead to ambiguity-averse agents being important in the short run and allow us to think jointly about heterogeneous portfolio choice and asset prices in a very tractable way. Clearly some of the features in the papers just reviewed can potentially be adopted here for a richer set of implications. With a different approach still, Yan (2008) deals with irrational investors whose preferences are not all the same and finds that it is possible for an irrational investor to dominate the market even if his beliefs deviate from the truth persistently and substantially.

3. The economy

This is an infinite-horizon exchange economy. Production is exogenous: the economy has a tree that pays dividends every period. The dividend grows at a random rate, which has a two-state support given by (λ_1, λ_2) and follows a first-order Markov process. The transition probabilities are given by $\phi_{ss'}$ - the probability of going to state s' if today's state is s, with $s, s' = 1,2$.

When the consumer is ambiguous about these probabilities, he perceives them to be

$$\Phi(v) = \begin{pmatrix} \phi_{11} - v_1 & \phi_{12} + v_1 \\ \phi_{21} - v_2 & \phi_{22} + v_2 \end{pmatrix},$$

where $v_s \in [-a, a](s = 1,2)$ with restrictions on a such that all probabilities are in $[0,1]$. The parameter a measures the amount of ambiguity in the economy.

Preferences are given by the maxmin formulation

$$V_t(s^t) = u[c(s^t)] + \beta \min_{\pi \in \Pi_{s^t}} E_\pi V_{t+1}(s^{t+1}),$$

where c is consumption, $u(c)$ is the period utility function, and Π_{s^t} is a set of transition probability laws given the history s^t today.

Aversion to ambiguity is captured by the "minimization" part in the utility formulation above: the consumer behaves with pessimism, i.e., he assumes the worst possible probability distribution. For an axiomatic foundation for this preference formulation, see Gilboa and Schmeidler (1989) for the static setting and Epstein and Wang (1994) and Epstein and Schneider (2003) for a multiperiod setting.

In the next section, we consider two types of agents whose ambiguity aversions differ. Consumer of type 1 is ambiguity averse and consumer of type 2 is not ambiguity averse $(v_2 = a_2 = 0)$. We focus on the case with *iid* and symmetric shocks.

4. Heterogeneity in ambiguity aversion

4.1. The decentralized economy

Markets are complete and consumers trade in equity shares of the tree and in a riskless bond that is in zero net supply. We denote the consumer's bond and equity holdings *b* and *e*, respectively.

We assume that $\lambda_1 > \lambda_2$ so that the bad outcome is state 2 – the outcome where the rate of growth of the dividend is lower. The ambiguity-averse consumer puts a higher weight on the bad outcome than what is warranted by the objective probability; that is, he becomes pessimistic because he is worried about that outcome and does not know its probability. The objective probability of this state is $1 - \phi$. His belief, however, is $1 - \phi + v$ and he chooses v from the set $v \in [-a, a]$. The higher is a, the more ambiguity there is in the economy. In what follows we define a competitive equilibrium in this economy and, afterwards, derive the demand for equity and bonds for the representative consumer and determine asset prices. The strategy here is to take first-order conditions from the representative agent's

[4] For a survey on the equity premium puzzle, see Kocherlakota (1996).

problem and use the consumption allocations derived in the planning problem (shown in the Appendix) as a way to impose market clearing.

We consider agents with a logarithmic period utility function.

4.1.1. Equilibrium definition

The consumer's problem is given recursively by

$$V(d,w,\theta) = \max_{c,b,e}\left\{\log c + \beta \min_{v\in[-a,a]} \sum_{s'=1}^{2} \phi_{s'}(v) V\left(\lambda_{s'}d, w'_{s'}, \theta'_{s'}\right)\right\},$$

subject to the budget constraint

$$c + p(d,\theta)e + q(d,\theta)b = w,$$
$$w'_{s'} = b + e\left[\lambda_{s'}d + p\left(\lambda_{s'}d, \theta'_{s'}\right)\right],$$

and the law of motion for $\theta'_{s'}$ given by

$$\theta'_{s'} = g_{s'}(d,\theta),$$

where (d,w,θ) is the state vector, d is today's dividend, w is the consumer's wealth today, θ is the relative wealth of type 1 consumer, p is the price of equity, e is the fraction of the equity share held by the consumer, q is the price today of a bond that pays one unit of the consumption good next period, and b is the holdings of the bond. (The argument d is included for g only for completeness; it will not be there under the log utility assumption.)

A recursive competitive equilibrium is a set of functions $V_i(d,w,\theta), c_i(d,w,\theta), b_i(d,w,\theta), e_i(d,w,\theta), v_i(d,w,\theta)$ for $i\in\{1,2\}$, with the following properties: (i) Consumer maximization: for each agent $i, V_i, c_i, b_i, e_i, v_i$ solve the dynamic programing problem above. (ii) Market clearing: for all values of the arguments and given that total wealth in the economy when the state variable is (d,θ) is $d + p(d,\theta)$:

$$c_1(d,\theta[d+p(d,\theta)],\theta) + c_2(d,(1-\theta)[d+p(d,\theta)],\theta) = d,$$
$$b_1(d,\theta[d+p(d,\theta)],\theta) + b_2(d,(1-\theta)[d+p(d,\theta)],\theta) = 0,$$

and

$$e_1(d,\theta[d+p(d,\theta)],\theta) + e_2(d,(1-\theta)[d+p(d,\theta)],\theta) = 1.$$

(iii) The relative wealth dynamics, finally, is given by

$$g_{s'}(d,\theta) = \frac{w'_{1s'}(d,\theta)}{w'_{1s'}(d,\theta) + w'_{2s'}(d,\theta)},$$

where

$$w'_{1s'}(d,\theta) \equiv b_1(d,\theta[d+p(d,\theta)],\theta) + e_1(d,\theta[d+p(d,\theta)],\theta)(d\lambda_{s'} + p[d\lambda_{s'}, g_{s'}(d,\theta)]),$$

and

$$w'_{2s'}(d,\theta) \equiv b_2(d,(1-\theta)[d+p(d,\theta)],\theta) + e_2(d,(1-\theta)[d+p(d,\theta)],\theta)(d\lambda_{s'} + p[d\lambda_{s'}, g_{s'}(d,\theta)]).$$

4.1.2. Asset allocation and price determination

Now we will show how to find prices and portfolio allocations in this economy. We take first-order conditions from the consumer's problem with respect to b and e. Then we use the planning problem and identify the θ in that problem with the corresponding variable here: the planning weight on agent 1 equals the relative fraction of total wealth held in equilibrium by this agent[5].

The Euler equation for bonds is given by

$$\frac{q(d,\theta)}{c_i(d,w,\theta)} = \beta\left((\phi - v)\frac{1}{c_i(\lambda_1 d, w'_{i1}, \theta'_1)} + (1-\phi + v)\frac{1}{c_i(\lambda_2 d, w'_{i2}, \theta'_2)}\right).$$

The solution to the planner's problem implied: $c_1(d,w,\theta) = \theta d$ (and $c_2(d,w,\theta) = (1-\theta)d$). We substitute these values in the Euler equation above, which implies that we are imposing market clearing.

Thus, we have

$$q(d,\theta) = \beta\theta\left(\frac{\phi - v}{\theta'_1\lambda_1} + \frac{1-\phi+v}{\theta'_2\lambda_2}\right).$$

Finally, substituting the values of θ' in each of the states (equations (1) and (2) below), which were solved for in the planner's problem, the price of bonds, $q(d,\theta)$, then becomes

$$q(d,\theta) = \beta\left[\frac{\phi}{\lambda_1} + \frac{1-\phi}{\lambda_2} + \theta v\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\right] \equiv \hat{q}(\theta).$$

The price of the bond is increasing in v. In addition, here, the price of bonds is increasing in θ. We show below that the ambiguity-averse agents demand the bond. The proceeds from the bond are more valuable when marginal utility of consumption is high (which occurs in the bad state) and the ambiguity-averse agents assign a higher probability to that state. As θ increases, there is a higher demand for the bond, so its price goes up.

[5] A description of the planning problem is presented in the Appendix (section A1).

Similarly, the price of equity, $p(d,\theta)$, is given by

$$p(d,\theta)=\beta\left\{\frac{(\phi-v)[\lambda_1 d+p(\lambda_1 d,\theta_1')]}{\lambda_1}+\frac{(1-\phi+v)[\lambda_2 d+p(\lambda_2 d,\theta_2')]}{\lambda_2}\right\},$$

where we recall that

$$\theta_1'=\frac{\theta(\phi-v)}{\phi-\theta v}\le\theta, \quad [1]$$

and

$$\theta_2'=\frac{\theta(1-\phi+v)}{1-\phi+\theta v}\ge\theta \quad [2]$$

from the planning problem. (The inequalities above follow since $v\ge 0$.)

The latter laws of motion reveal that the ambiguity-averse agent gains in relative wealth when the state is bad and loses when it is good: his probability "beliefs" are tilted toward the bad state.

We see that $p(d,\theta)=d\hat{p}(\theta)$ solves this equation, delivering

$$\hat{p}(\theta)=\beta\{(\phi-v)[1+\hat{p}(\theta_1')]+(1-\phi+v)[1+\hat{p}(\theta_2')]\}.$$

This is a functional equation: it holds for all θ (recall that v may also depend on θ). The solution to this functional equation is

$$\hat{p}(\theta)=\frac{\beta}{1-\beta},$$

and

$$p(d,\theta)=d\frac{\beta}{1-\beta}.$$

So the price of equity does not depend on θ. This is because the utility function is logarithmic.

The equilibrium holdings of equity of consumer 1, which can be obtained by using the expression for future wealth, $w_{1s'}'=b_1+e_1(\lambda_{s'}d+p_{s'}')$, together with the equilibrium condition that $w_{1s'}'=\theta_{s'}'(d\lambda_{s'}+p_{s'}')$, are given by

$$e_1(d,\theta)=\frac{\theta_1'\lambda_1-\theta_2'\lambda_2}{\lambda_1-\lambda_2}\equiv\hat{e}_1(\theta).$$

Thus, the equity holdings of agent 1 are independent of the level of d. We see that if $v=0$, in which case $\theta_1'=\theta_2'=\theta$, then $\hat{e}_1(\theta)=\theta$: the consumer's share of the tree equals his initial share of total wealth.

On the other hand, when $v>0$ (recall that without loss of generality we use $\lambda_1>\lambda_2$), we know that $\theta_1'<\theta<\theta_2'$, which makes the holdings of equity lower as compared to the case when $v=0$. That is, the ambiguity-averse agent will have a smaller share of equity holdings than his overall wealth would otherwise prescribe: this is a portfolio composition effect. How much his portfolio composition will be changed must be numerically examined.

We can also examine the portfolio effect by looking at the amount of bonds purchased by agent 1. His equilibrium holdings of bonds are obtained as

$$b_1(d,\theta)=\frac{d\lambda_1}{1-\beta}[\theta_1'-\hat{e}_1(\theta)],$$

It is interesting to note here that bond holdings are proportional to d. This is because type 1 agents' consumption is given by θd and they mainly hold bonds. Naturally, they are zero in the special case $v=0$, when $e_1=\theta$ and $\theta'=\theta$. Moreover,

$$\theta_1'-\hat{e}_1(\theta)=\theta_1'-\frac{\theta_1'\lambda_1-\theta_2'\lambda_2}{\lambda_1-\lambda_2}=$$

$$=\theta_1'\left(1-\frac{\lambda_1-\frac{\theta_2'}{\theta_1'}\lambda_2}{\lambda_1-\lambda_2}\right)>0,$$

since $\theta_2'>\theta_1'$, and thus we conclude, consistently with the above insights regarding equity holdings, that the ambiguity-averse agent increases his bond holdings relative to the $v=0$ zero-bonds case: his portfolio composition moves away from equity and into bonds because he is more pessimistic than person 2 in his perception of the return (performance) of equity.

The ambiguity-averse consumer is exposed to two sources of uncertainty in this economy: (i) the payoff of equity and (ii) the price of the bond. The price of the bond depends on θ, the relative wealth of consumer 1, and this variable is random. In particular, since $\theta_2'>\theta>\theta_1'$, the price of the bond, q, increases if state 2 occurs and it decreases if state 1 occurs.

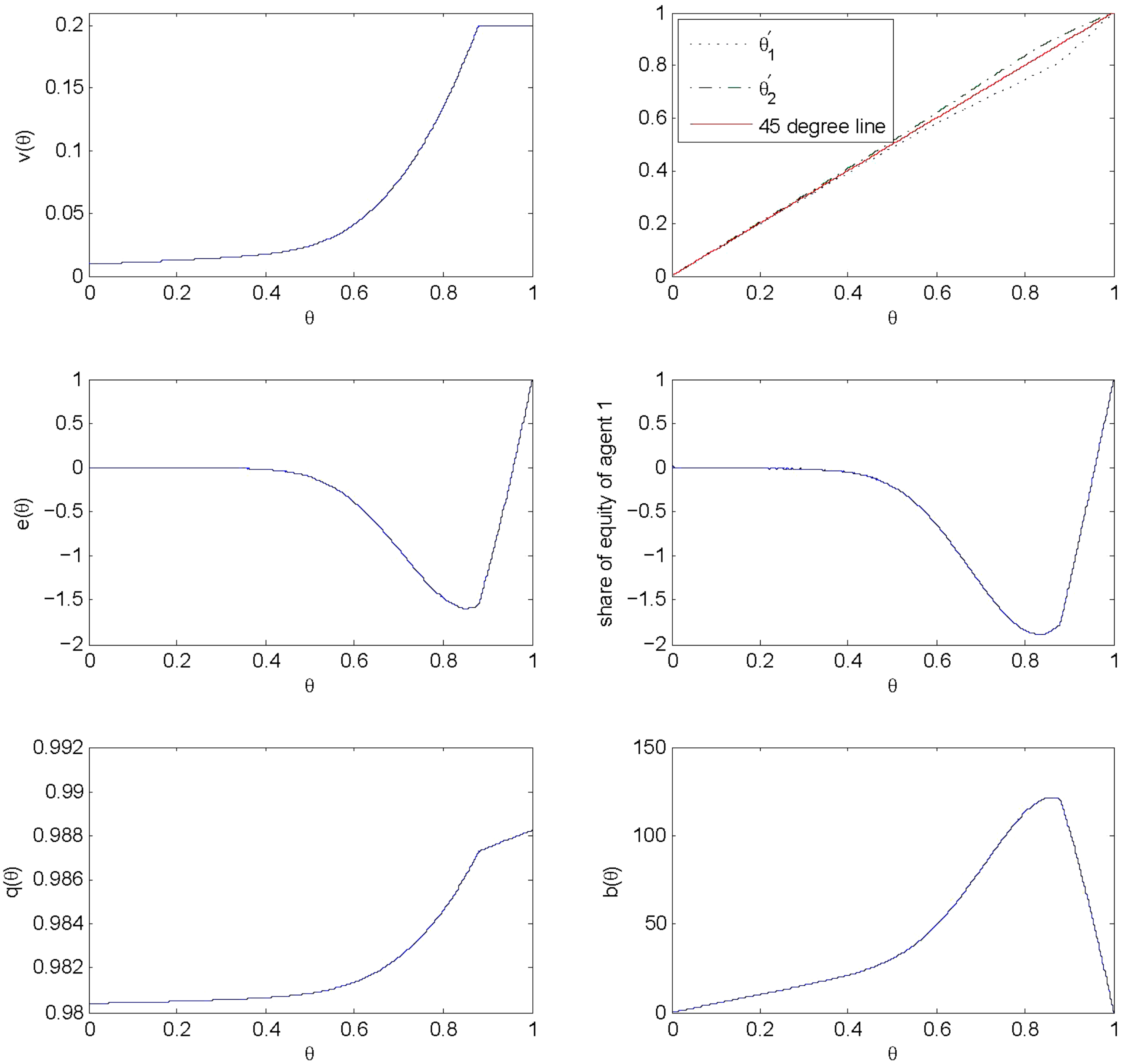
Below we numerically compute solutions for $v(\theta),\theta_1'(\theta),\theta_2'(\theta),e(\theta),\frac{p(\theta)e(\theta)}{p(\theta)e(\theta)+q(\theta)b(\theta)},q(\theta)$, and $b(\theta)$ for agent 1.

4.1.3. Numerical example

The parameter values are $\lambda_1=1.02,\lambda_2=0.98,\beta=0.98,\phi=0.5,d=1$, and $a=0.2$.

As we see from the graphs in figure 1, the ambiguity-averse consumer short-sells equity for most values of θ. The reason for this is the following. State 2 is bad for the ambiguityaverse consumer for two reasons: (i) the payoff from equity is low and (ii) the price of the bond increases so that it makes the good next period more expensive (this consumer does not own any goods next period). Therefore, to provide protection against the former type of uncertainty, the ambiguity-averse consumer buys bonds and to provide protection against the latter type of uncertainty, the ambiguity-averse consumer sells equity short. We shall analyze the details of the graphs in figure 1 in what follows.

Figure 1: From left to right from top to bottom: (a) $v(\theta)$, (b) law of motion for θ (θ_1 below the 45 degree line; θ_2 above the 45 degree line), (c) $e(\theta)$ = equity holdings for agent 1, (d) share of equity of agent 1, (e) $q(\theta)$, (f) $b(\theta)$ = bond holdings for agent 1



Outcomes in the two extreme cases: $\theta = 0$ and $\theta = 1$:

When θ is 0 - the standard agent has all the wealth - asset pricing is standard because all assets are priced by the standard agent. In this case, there is voluntary non-participation in equity: at $\theta = 0$, a measure-zero ambiguity-averse agent chooses not to participate in equity markets. To see this, let $v^* > 0$ be the amount of pessimism such that the agent holds only bonds and zero equity. If he had positive equity, his portfolio would do well in good times, so he would be very pessimistic with regard to that state choosing $v = a$. But such belief would imply that selling equity short is optimal which is a contradiction. Similarly, negative equity implies $v = -a$, which implies that positive equity is optimal which again is a contradiction. So the only possibility for the ambiguity-averse agent at $\theta = 0$ is to hold zero equity, hold bonds only, choose $0 < v^* < a$, and have constant consumption. In this case there is no uncertainty from a change in the price of bonds next period since $\theta_1' = \theta_2' = 0$. We show in the Appendix an analytical solution for v^*: $v^* = \phi - \frac{\lambda_2}{\lambda_1+\lambda_2}$. For the parameter values considered above, $v^* = 0.00246$.

When θ is 1 - the ambiguity-averse agent has all the wealth - asset pricing is non-standard. In particular, there is a high equity premium: since the representative agent, who must be holding all the equity and no bonds, now is ambiguity-averse, v must be positive and high (at its bound, if a is small). This means pessimism toward the good state, a very high bond price, and a low risk-free rate. Thus, the equity premium becomes very high. With constant relative risk-aversion utility function and more than log curvature, the price of equity would fall and the equity premium would be even higher. The case with a representative agent with ambiguity aversion is treated in the Appendix.

Intermediate values: $\theta > 0$ but not too high

Unless θ is quite high (in the numerical example), the ambiguity-averse agent is still at an interior solution: the present value utility next period is the same for each value of λ.

When θ is positive but not too large, the ambiguity-averse agent short-sells the stock. Why? State 2 (the bad state) is now bad in the sense that, since θ rises (equity does poorly, so the standard agent loses in relative terms), the price of the bond goes up since ambiguityaverse agents become richer in relative terms and demand more of it.

A very small amount of pessimism rationalizes their choice, i.e., liking state 2 makes you sell equity short. Thus, you are still indifferent between which state occurs: if state 1 occurs, the bond price is low (good, since they demand it); if state 2 occurs, their portfolio does well. v is set so that the present value utility is equal across the two outcomes.

Intermediate values: nonlinearity

If θ is positive but small, changes in θ do not have any considerable effects on q, so the randomness in q is not so important. Then, ambiguity-averse consumers mainly hold bonds and short-sell equity somewhat to protect against the uncertainty in q. This asset choice makes

$$V(w_1', \theta_1') = V(w_2', \theta_2')$$

for a small value of the belief v; that is, v is still an interior solution.

The higher is θ, the higher is the demand for bonds. Therefore, θ_1' and θ_2' are more spread out than before because the two types of consumers take very different asset positions. Thus, the swings in the price of the bond are higher so the ambiguity averse agents buy bonds and short sell equity more heavily, and an even higher v is needed to justify this behavior. To understand this in somewhat more detail, note that the value of v is larger, reflecting more pessimism about state 2. Since V is decreasing in θ and increasing in w (the former is true because q is increasing in θ) and since θ_2' is much larger than θ_1', w_2' needs to be much larger than w_1' in order to equate $V(w_1', \theta_1')$ and $V(w_2', \theta_2')$ - and hence still make v an interior solution. This is achieved by short-selling equity even more heavily. Thus, a high θ and a high v are reinforcing; as can be seen, θ and v appear multiplicatively in the bond-price formula above! I.e., we have a nonlinearity in the portfolio pattern. So when θ is high, ambiguity aversion makes the fluctuations in the price of the bond very large.

At some point, v hits the bound a. The strategy of buying more bonds and short selling more heavily to protect against risk next period stops working since pessimism has reached its highest value. Therefore, present value utilities next period are not equalized. Beyond this bound of pessimism, as the ambiguity-averse agent has a higher fraction of total wealth, he will decrease the bond holding and increase the equity holding, simply for market-clearing reasons (recall: at $\theta = 1$ he holds only equity). Over this range, the equity premium increases sharply, to motivate the ambiguity-averse agent to hold more equity and less bonds. Close to $\theta = 1$, the fluctuations in θ have become very small, and the uncertainty resulting from changes in q is therefore also very small and ambiguity-averse agents consequently do not need to short-sell the stock.

4.2. Relative consumption and wealth in the long run

We can show analytically[6] that

$$E(\theta' \mid \theta) < \theta,$$

i.e., that over time, the relative wealth of the ambiguity-averse agents decreases toward zero: these agents disappear, economically speaking[7].

However, it can also be shown that

$$E\left(\frac{\theta'}{\theta}\right) \to_{\theta \to 0} 1,$$

so the rate at which they disappear goes to zero: they remain with positive wealth for a long, long time.

[6] The proofs of expressions (23) and (24) are in the Appendix. This result and the following discussion are reminiscent of the analysis in Coen-Pirani (2004).

[7] The result is similar to the one in Sandroni (2000), in a set-up where agents have different abilities to make accurate belief predictions.

5. Conclusion

In this paper, we have studied a very simple and tractable asset pricing model and by exploring an economy where some consumers are ambiguity-averse and others are not, we find that, by making consistently "bad bets", ambiguity-averse consumers will see their relative wealth decline over time, and thereby asset prices will be increasingly dominated by standard consumers. Note also that these bad bets are not bad in the sense of "crazy portfolios", but simply in the sense of delivering a lower return on average by not investing enough in stock. In particular, the ambiguity-averse consumers eventually choose not to participate at all in the stock market: the other, standard consumers hold all risk (and get all the high returns on average).

It is interesting to note that there is (close to) non-participation for a large range of values for θ. Thus, without having to assume that there are costs of transacting/investing in stock, we can use this setting to derive conditions under which a large fraction of the population—the ambiguity-averse—(almost) do not have any stock. This kind of result was also derived in Epstein and Schneider's (2008) work. Unless the ambiguity averse consumers are measure zero, exact non-participation cannot be obtained here because the risk-free rate fluctuates with the endowment shock; because the ambiguity-averse agents hold bonds, it is optimal for them to use equity to hedge against the interest-rate risk. This risk, however, is very small for a large range of (low) values of θ: when θ is zero, the risk-free rate is constant, and thus not until the ambiguity-averse agents have a significant fraction of total wealth will these fluctuations be large enough to induce significant equity holdings for these agents.

To make the wealth distribution not converge to an extreme outcome, one could consider an overlapping-generations structure, where in each generation of newborns with zero debt, some are ambiguity-averse; that way, a significant part of aggregate wealth will always belong to ambiguity-averse consumers.

Finally, the implications for wealth inequality in this class of models is very striking and deserve further study. The observed inequality in wealth is difficult to explain with standard versions of calibrated macroeconomic models, and our present work seems to be one avenue for understanding why some consumers are so rich and others so poor (see, e.g., Huggett, 1993, Aiyagari, 1994, Krusell and Smith, 1998, and Piketty, 2014). It should also be noted in this context that our model has rather extreme long-run implications (both for asset prices and for wealth) to a large extent because (i) agents live forever and (ii) asset markets are complete. It should prove particularly fruitful to examine versions that relax one or both of these assumptions.

References

Aiyagari, S. Rao (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving", The Quarterly Journal of Economics, 109(3), pp. 659-684.

Angeletos, George-Marios and Jennifer La'O (2009): "Noisy Business Cycles", NBER Macroeconomics Annual, 2009.

Alonso, Irasema and Roman Sysuyev (2009): "Ambiguity in a Two-Country World", Columbia University.

Alonso, Irasema and Mauricio Prado (2015): "Ambiguity Aversion, Asset Prices, and the Welfare Costs of Aggregate Fluctuations", Journal of Economic Dynamics and Control, 51,79-92.

Borovička, Jaroslav (2015): "Survival and Long-Run Dynamics with Heterogeneous Beliefs under Recursive Preferences", mimeo, New York University.

Coen-Pirani, Daniele (2004): "Effects of Differences in Risk Aversion on the Distribution of Wealth", Macroeconomic Dynamics, 8, 617-632.

Colacito, Riccardo and Mariano Massimiliano Croce (2014): "Recursive Allocations and Wealth Distribution with Multiple Goods: Existence, Survivorship, and Dynamics", mimeo, University of North Carolina, Chapel Hill.

Condie, Scott (2008): "Living with Ambiguity: Prices and Survival when Investors Have Heterogeneous Preferences for Ambiguity", Economic Theory, 36, 81-108.

Cvitanić, Jakša and Semyon Malamud (2011): "Price Impact and Portfolio Impact", Journal of Financial Economics, 100, 201-225.

Epstein, Larry G. and Jianjun Miao (2003): "A Two-person Dynamic Equilibrium under Ambiguity", Journal of Economic Dynamics and Control, 27, 1253-1288.

Epstein, Larry G. and Martin Schneider (2003): "Recursive Multiple-Priors", Journal of Economic Theory, 113, 1-31.

Epstein, Larry G. and Martin Schneider (2007): "Learning under Ambiguity", Review of Economic Studies, 74, 1275-1303.

Epstein, Larry G. and Martin Schneider (2008): "Ambiguity, Information Quality and Asset Pricing", Journal of Finance, 63, 197-228.

Epstein, Larry G. and Tan Wang (1994): "Intertemporal Asset Pricing under Knightian Uncertainty," Econometrica 62, 283-322.

Fostel, Ana and John Geanakoplos (2008): "Leverage Cycles and the Anxious Economy, American Economic Review, 98, 1211-1244.

Gilboa, Itzhak and David Schmeidler (1989): "Subjective Probability and Expected Utility Without Additivity", Econometrica, 57, 571-587.

Guerdjikova, Ani and Emanuela Sciubba (2015): "Survival with Ambiguity", Journal of Economic Theory, 155, 50-94.

[Hatchondo, Juan Carlos (2008): "Asymmetric Information and the Lack of Portfolio Diversification", International Economic Review 49(4), 1297-1330.

Hatchondo, Juan Carlos, Per Krusell and Martin Schneider (2014): "Asset Trading and Valuation with Uncertain Exposure", working paper, Federal Reserve Bank of Richmond, IIES, and Stanford University.

Huggett, Mark (1993), "The risk-free rate in heterogeneous-agent incomplete-insurance economies", Journal of Economic Dynamics and Control, 17(5-6), 953-969.

Kocherlakota, Narayana R. (1996): "The Equity Premium: It's Still A Puzzle", Journal of Economic Literature, 34, 42-71.

Krusell, Per and Anthony A. Smith, Jr. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", Journal of Political Economy 106:5, 867-896.

Lucas, Robert E. Jr. (2003): "Macroeconomic Priorities", American Economic Review, 93,1-14.

Lucas, Robert E. Jr. and Nancy Stokey (1984): "Optimal Growth with Many Consumers", Journal of Economic Theory, 32, 139-171.

Mehra, Rajnish and Edward C. Prescott (1985): "The Equity Premium: A Puzzle", Journal of Monetary Economics, 15, 145-161.

Piketty, Thomas (2014): Capital in the Twenty-First Century, The Belknap Press of Harvard University Press.

Sandroni, Alvaro (2000): "Do Markets Favor Agents Able to Make Accurate Predictions?"', Econometrica, 68, 1303-1341.

Yan, Hongjun (2008): "Natural Selection in Financial Markets: Does It Work?", Management Science, 54, 1935-1950.

Appendix

A1. The planner's problem

The state vector is (d, θ, s): today's dividend, the weight the planner puts on consumer 1, and today's shock. The planner solves the problem

$$V_s(d,\theta) = \max_{c_1,c_2,z_{1s'},z_{2s'}} \theta \log c_1 + (1-\theta)\log c_2 + \\ +\beta \left\{ \begin{array}{c} \min_{v_1 \in [-a_1,a_1]} \theta \sum_{s'=1}^{2} \phi_{ss'}(v_1) z_{1s'} + \\ \min_{v_2 \in [-a_2,a_2]} (1-\theta) \sum_{s'=1}^{2} \phi_{ss'}(v_2) z_{2s'} \end{array} \right\}$$

subject to

$$\min_{\theta'_{s'}} V_{s'}(d\lambda_{s'}, \theta'_{s'}) - \left[\theta'_{s'} z_{1s'} + (1-\theta'_{s'}) z_{2s'}\right] \geq 0, \qquad [3]$$

and

$$c_1 + c_2 = d, \qquad [4]$$

where c_i is agent i 's consumption, $i = 1,2, z_i$ is next period's present-value utility for agent $i, \phi_{s1}(v_i) = \phi_{s1} - v_i$, and $\phi_{s2}(v_i) = \phi_{s2} + v_i$. The first constraint (25) makes the problem recursive and the second constraint (26) is the resource constraint. This formulation which is based on Lucas and Stokey (1984) is also used in Alonso and Sysuyev (2009).

Taking first-order conditions with respect to the consumption of agents 1 and 2, we have

$$c_1 = \theta d,$$

and

$$c_2 = (1-\theta) d,$$

with respect to $z_1(1)$ and $z_2(1)$, we obtain

$$\frac{\theta'_1}{1-\theta'_1} = \frac{\theta(\phi_{s1} - v_1)}{(1-\theta)(\phi_{s1} - v_2)}, \qquad [5]$$

and similarly with respect to $z_1(2)$ and $z_2(2)$, we have

$$\frac{\theta'_2}{1-\theta'_2} = \frac{\theta(\phi_{s2} + v_1)}{(1-\theta)(\phi_{s2} + v_2)}. \qquad [6]$$

After some algebra where we use the law of motion for $\theta'(s)$ (equations 29 and 30), we can rewrite the planner's problem as

$$V_s(d,\theta) = \max_{c_1,c_2} \theta \log c_1 + (1-\theta)\log c_2 + \\ +\beta \min_{v_1,v_2} \left\{ \textstyle\sum_{s'=1}^{2} \phi_{ss'}(\theta v_1 + (1-\theta) v_2) V_{s'}(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}(\theta v_1 + (1-\theta) v_2)},$$

and

$$c_1 + c_2 = d.$$

Note that $\phi_{s1}(\theta v_1 + (1-\theta)v_2) = \phi_{s1} - \theta v_1 - (1-\theta)v_2$ and $\phi_{s2}(\theta v_1 + (1-\theta)v_2) = \phi_{s2} + \theta v_1 + (1-\theta)v_2$.

In the case where shocks are *iid* and symmetric and consumer 2 is not ambiguity-averse ($v_2 = a_2 = 0$), the planner's problem becomes[8]:

$$V(d,\theta) = \max_{c_1,c_2,\theta'_{s'}} \theta \log c_1 + (1-\theta)\log c_2 + \\ \beta \min_{v \in [-a,a]} \left\{ \textstyle\sum_{s'=1}^{2} \phi_{s'}(\theta v) V(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{s'}(v)}{\phi_{s'}(\theta v)},$$

and

$$c_1 + c_2 = d.$$

Using the FOCs for consumption, we obtain $c_1 = \theta d$ and $c_2 = (1-\theta)d$, so we obtain

$$V(d,\theta) = \log d + \log \theta^\theta (1-\theta)^{1-\theta} \\ + \beta \min_v [(\phi - \theta v) V(d\lambda_1, \theta'_1) \\ + (1 - \phi + \theta v) V(d\lambda_2, \theta'_2)],$$

with

$$\theta'_1 = \theta \frac{\phi - v}{\phi - \theta v},$$

and

$$\theta'_2 = \theta \frac{1 - \phi + v}{1 - \phi + \theta v}.$$

Here, we conjecture that $V(d,\theta)$ takes the form $A \log d + W(\theta)$. This guess delivers

$$A \log d + W(\theta) = \log d + \log \theta^\theta (1-\theta)^{1-\theta} + \\ +\beta \min_v \left\{ \begin{array}{c} (\phi - \theta v)[A\log(d\lambda_1) + W(\theta'_1)] + \\ (1 - \phi + \theta v)[A \log(d\lambda_2) + W(\theta'_2)] \end{array} \right\}.$$

Inspecting this functional equation, it can be seen that $A = \frac{1}{1-\beta}$ works and we can express $W(\theta)$ as

$$W(\theta) = \log \theta^\theta (1-\theta)^{1-\theta} + \\ +\beta \min_{v \in [-a,a]} \left\{ \begin{array}{c} (\phi - \theta v)\left[\frac{\log \lambda_1}{1-\beta} + W\left(\theta \frac{\phi - v}{\phi - \theta v}\right)\right] \\ +(1 - \phi + \theta v)\left[\frac{\log \lambda_2}{1-\beta} + W\left(\theta \frac{1 - \phi + v}{1 - \phi + \theta v}\right)\right] \end{array} \right\}.$$

This is a one-dimensional dynamic programming problem delivering optimal v as a function of θ and hence, a law of motion for θ. The variable θ also corresponds to the fraction of the total wealth - the current dividend plus the value of the tree - owned by agent 1 in a complete-markets equilibrium. The following figures for $W(\theta)$ and $v(\theta)$

[8] From now on, we drop the subscript on v and a, since it should be clear that they refer only to consumer 1.

below assume the following values for the parameters: $\lambda_1 = 1.02, \lambda_2 = 0.98, \beta = 0.98, \phi = 0.5$, and $a = 0.2$.

Figure 2 graphs $W(\theta)$ for $a = 0$ and $a = 0.2$. $W(\theta)$ reveals a shape similar to $\log \theta^{\theta}(1- \theta)^{1-\theta}$, which is the (constant) flow utility of a planner in a two-type economy where no consumer has ambiguity aversion: the planner has to give consumption to two types of agents and the period utility is logarithmic. When $a = 0.2, W(0)$ is bigger than $W(1)$ since present value utility when all agents are not ambiguity averse (which occurs when $\theta = 0$) is higher than when all agents are ambiguity averse (which occurs when $\theta = 1$).

Figure 2: Value function $W(\theta)$

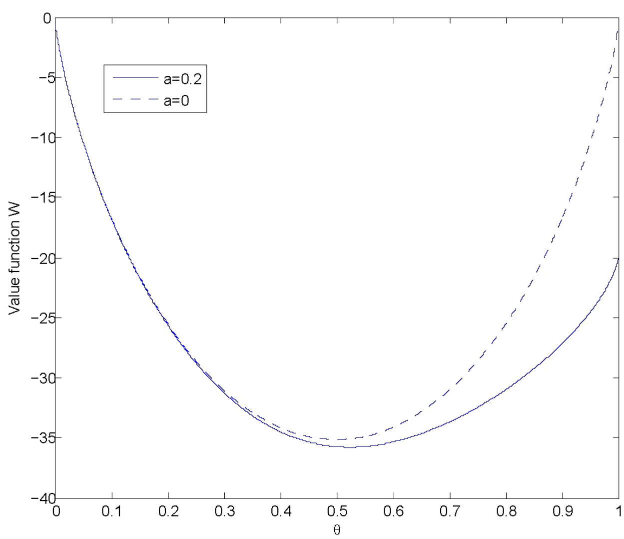
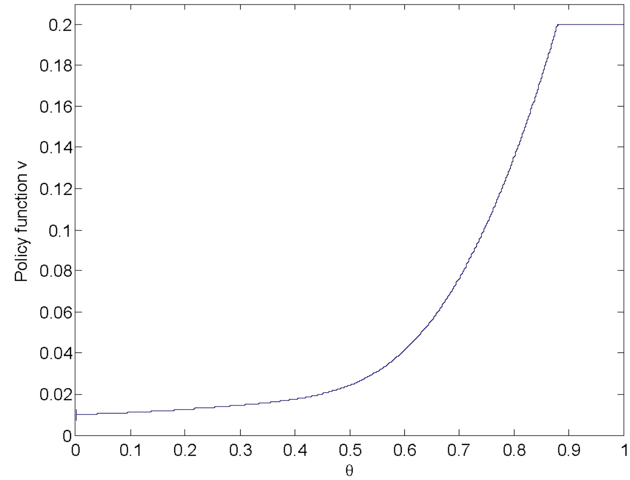


Figure 3, for the optimal choice of v, shows that v is close to zero and interior at first (for small θ's), and then it increases monotonically in θ and reaches the upper bound *a* for a value of θ a little below 0.9. These findings were interpreted in section 4.1.3.

Figure 3: Policy function $v(\theta)$



A2. The special case where $\boldsymbol{\theta = 0}$

We solve the problem for an ambiguity-averse agent who is measure zero in the economy.

This agent solves the problem

$$V(w,d) = \max_{c,b,e} u(c) + \min_{v} \beta[(\phi - v)V(w_1', d\lambda_1) + (1-\phi + v)V(w_2', d\lambda_2)]$$

subject to

$$c + qb + pde = w]$$
$$w_1' = b + (\lambda_1 d + p\lambda_1 d)e$$
$$w_2' = b + (\lambda_2 d + p\lambda_2 d)e$$

The FOCs with respect to *b* are

$$qu'(w - qb + pde) = \beta\{(\phi - v)u'[b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'] + (1-\phi + v)u'[b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e']\}$$

and with respect to *e*, they are

$$pdu'(w - qb + pde) = \beta\{(\phi - v)u'[b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e']\lambda_1 d(1+p) + (1-\phi + v)u'[b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e']\lambda_2 d(1+p)\}$$

Using logarithmic utility, we see that these equations become

$$q = \beta\left[\frac{\phi - v}{b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'} + \frac{1-\phi+v}{b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'}\right](w - qb + pde)$$

$$\frac{p}{1+p} = \beta\left[\frac{(\phi - v)\lambda_1}{b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'} + \frac{(1-\phi+v)\lambda_2}{b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'}\right](w - qb + pde)$$

We guess that

$b = \alpha_b w$
and
$ed = \alpha_e w.$

Then

$$q = \beta\left\{\frac{\phi - v}{[\alpha_b + \alpha_e\lambda_1(1+p)](1 - q\alpha_b - p\lambda_1\alpha_e)} + \frac{1-\phi+v}{[\alpha_b + \alpha_e\lambda_2(1+p)](1 - q\alpha_b - p\lambda_2\alpha_e)}\right\}(1 - q\alpha_b - p\alpha_e)$$

$$\frac{p}{1+p} = \beta\left\{\frac{(\phi - v)\lambda_1}{[\alpha_b + \alpha_e\lambda_1(1+p)](1 - q\alpha_b - p\lambda_1\alpha_e)} + \frac{(1-\phi+v)\lambda_2}{[\alpha_b + \alpha_e\lambda_2(1+p)](1 - q\alpha_b - p\lambda_2\alpha_e)}\right\}(1 - q\alpha_b - p\alpha_e)$$

The problem of the consumer can be rewritten as

$$V(w) = \max_{c,b,e}\log c + \min_{v}\beta[(\phi - v)V(w_1') + (1 - \phi + v)V(w_2')]$$

subject to

$c + qb + p\hat{e} = w$
$w_1' = b + \hat{e}\lambda_1(1+p)$
$w_2' = b + \hat{e}\lambda_2(1+p),$

where $\hat{e} \equiv de$. The variable v will be chosen (due to the envelope theorem) so that $V(w_1') = V(w_2')$, i.e., so that $w_1' = w_2'$. That means that $b = \alpha_b w$ and $\hat{e} = 0$ - the agent does not hold equity - and from the FOC above, that

$$q = \beta\left[\frac{\phi - v}{\alpha_b(1 - q\alpha_b)} + \frac{1-\phi+v}{\alpha_b(1 - q\alpha_b)}\right](1 - q\alpha_b).$$

This expression simplifies to

$\alpha_b q = \beta$

and then consumption is given by

$c = (1-\beta)w.$

Since $\frac{p}{1+p} = \beta$ in the $\theta = 1$ case, this implies

$\alpha_b = (\phi - v)\lambda_1 + (1 - \phi + v)\lambda_2.$

Therefore,

$$v = \frac{\phi\lambda_1 + (1-\phi)\lambda_2 - \alpha_b}{\lambda_1 - \lambda_2} = \phi - \frac{\alpha_b - \lambda_2}{\lambda_1 - \lambda_2}$$

When $\theta = 0, q = \beta\left(\frac{\phi}{\lambda_1} + \frac{(1-\phi)}{\lambda_2}\right)$. This means that for $\phi = 0.5, \alpha_b = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$

Then, for $\phi = 0.5, \beta = 0.98, \lambda_1 = 1.02$, and $\lambda_2 = 1.01$,

$v = \phi - \frac{\lambda_2}{\lambda_1+\lambda_2} = 0.00246.$

A3. Proofs of subsection 4.2

We want to prove that

$$E(\theta' \mid \theta) < \theta. \quad [7]$$

Since

$$E(\theta' \mid \theta) = \phi\frac{\theta(\phi - v)}{\phi - \theta v} + (1-\phi)\frac{\theta(1-\phi+v)}{1-\phi+\theta v},$$

expression (7) becomes:

$$\phi\frac{\theta(\phi - v)}{\phi - \theta v} + (1-\phi)\frac{\theta(1-\phi+v)}{1-\phi+\theta v} < \theta. \quad [8]$$

Simplifying (8) yields:

$\theta^2 v^2 < \theta v^2.$

Since $v \neq 0$,

$$\theta < 1. \quad [9]$$

And condition (9) is always true in the case we study, otherwise we would be back to the case of one agent.

The proof that $\lim_{\theta\to 0} E\left(\frac{\theta'}{\theta}\right) = 1$ is even simpler. First, consider the expression for $E\left(\frac{\theta'}{\theta}\right)$:

$$E\left(\frac{\theta'}{\theta}\right) = \phi\frac{\phi - v}{\phi - \theta v} + (1-\phi)\frac{1-\phi+v}{1-\phi+\theta v}.$$

Then, the limit becomes:

$$\lim_{\theta\to 0}\phi\frac{\phi - v}{\phi - \theta v} + (1-\phi)\frac{1-\phi+v}{1-\phi+\theta v} = \phi\frac{\phi - v}{\phi} + (1-\phi)\frac{1-\phi+v}{1-\phi}. = \phi - v + 1 - \phi + v = 1$$

A4. Representative-agent asset pricing

In this section and for simplicity, we first consider an ambiguity-averse representative agent with a logarithmic period utility function and discount factor β. In addition, we first assume that shocks are *iid* and symmetric, i.e., $\phi_{ss'} = 0.5$. After that, we consider a CRRA period utility function and assume serially correlated shocks.

There is an equity share that is competitively traded and a riskless bond that is in zero net supply. We denote the consumer's bond and equity holdings b and e, respectively.

The representative agent holds the tree and thus, his consumption in every period is the dividend of the tree. A log-period utility function together with the assumption of *iid* shocks imply that p, the price of the tree, will be linear in d, the dividend, and independent of the state: $p(d) = \hat{p}d$.

The ambiguity-averse consumer puts a higher weight on the bad outcome than what is warranted by the objective probability; that is, he becomes pessimistic because he is worried about that outcome and does not know its probability.

We assume that $\lambda_1 > \lambda_2$ so that the bad outcome is state $2 -$ the outcome where the dividend is low. The objective probability of this state is 0.5, but he chooses the belief in the bad state. His belief is $\phi(v) = 0.5 - v$ and he chooses v from the set $v \in [-a, a]$. The higher is a, the more ambiguity there is in the economy.

The problem of the representative agent with wealth today given by w is

$$V(w) = \max_{e} \log\,[w - p(d)e] + \beta \min_{v\in[-a,a]} (\phi - v)V(w_1') + (1 - \phi + v)V(w_2')$$

subject to

$$w_1' = [\lambda_1 d + p(d\lambda_1)]e,$$

and

$$w_2' = [\lambda_2 d + p(d\lambda_2)]e.$$

Here, for ease of notation, we have excluded the bond (since bond holdings must be zero in equilibrium). Moreover, the budget constraint: $c + p(d)e + q(d)b = w$ where $w = [d + p(d)]e_{-1} + b_{-1}(e_{-1}$ and b_{-1} are equity and bond holdings chosen in the previous period) has been substituted away. The Euler equation for equity is

$$p(d)u'(d) = \beta \begin{Bmatrix} (\phi - a)[\lambda_1 d + p(\lambda_1 d)]u'(\lambda_1 d) + \\ (1 - \phi + a)[\lambda_2 d + p(\lambda_2 d)]u'(\lambda_2 d) \end{Bmatrix}.$$

Clearly, p is linear in d (a constant times d), whenever $u'(c) = c^{-\sigma}$ (here, $\sigma = 1$). Since the period utility is logarithmic, the price of equity does not depend on beliefs because the payoff and the inverse of marginal utility (u') are proportional to λd so that the payoff times marginal utility is the same in both states. Thus, $p(d) = \frac{\beta}{1-\beta} d$ solves the Euler equation above: the price of equity is independent of ϕ and a.

Trivially here, since $e = 1$ in equilibrium, $w_1' = \frac{\lambda_1 d}{1-\beta}, w_2' = \frac{\lambda_2 d}{1-\beta}$, then $V(w_1') > V(w_2')$, so the solution for v is a corner, i.e., $v = a$. In section 4 we show that v can be an interior solution when the economy is populated by both ambiguity-averse and standard consumers.

The Euler equation for bonds similarly gives

$$q(d)u'(d) = \beta\{(\phi - a)u'(\lambda_1 d) + (1 - \phi + a)u'(\lambda_2 d)\}$$

We see that q depends on beliefs:

$$q = \beta\left[\left(\frac{\phi}{\lambda_1} + \frac{1-\phi}{\lambda_2}\right) + a\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\right].$$

The higher is a - the more ambiguity aversion there is in the economy - the higher is the belief that the bad state will happen, and the higher is the present value of one unit tomorrow, since the probability weight placed on the state with a high marginal utility is higher.

The net expected return on equity, ER_e, is given by

$$ER_e = \frac{\phi\lambda_1 + (1-\phi)\lambda_2}{\beta} - 1,$$

and it is independent of the belief. The net return on bonds, R_b, decreases when ambiguity aversion increases, because $R_b = \frac{1}{q} - 1$.

The equity premium in this economy is

$$ER_e - R_b = \frac{\phi\lambda_1 + (1-\phi)\lambda_2}{\beta} - \frac{\lambda_1\lambda_2}{\beta[(1-\phi)\lambda_1 + \phi\lambda_2 + a(\lambda_1 - \lambda_2)]}.$$

If we make $\phi = 0.5$, then the equity premium in this economy is

$$ER_e - R_b = \frac{\lambda_1 + \lambda_2}{2\beta} - \frac{\lambda_1\lambda_2}{\beta[0.5(\lambda_1 + \lambda_2) + a(\lambda_1 - \lambda_2)]}.$$

When ambiguity is most extreme, i.e., when $a = 0.5$, the equity premium becomes

$$\frac{\lambda_1 - \lambda_2}{2\beta}.$$

Using $\lambda_1 = 1.02, \lambda_2 = 1.01$, and $\beta = 0.98$, the equity premium is 0.5%, which is 200 times larger than the equity premium for the same parameter values when $a = 0$ - the standard model. Although this is an example, and not a calibration, it illustrates that the effect of ambiguity on asset prices/returns can be substantial.

If the period utility is $u(d) = \frac{d^{1-\alpha}}{1-\alpha}$, the price of equity depends on beliefs. In fact:

$$\hat{p} = \frac{\beta[(\phi - a)\lambda_1^{1-\alpha} + (1 - \phi + a)\lambda_2^{1-\alpha}]}{1 - \beta[(\phi - a)\lambda_1^{1-\alpha} + (1 - \phi + a)\lambda_2^{1-\alpha}]}.$$

A4.1 Serial correlation

We now assume that the period utility is $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$ and the shocks are serially correlated.

The problem of the representative agent with wealth today given by w and today's shock s is

$$V_s(w) = \max_e u[w - p_s(d)e] + \beta \min_{v_s \in [-a,a]} (\phi_{s1} - v_s)V_1(w_1') + (\phi_{s2} + v_s)V_2(w_2')$$

subject to

$w_1' = [\lambda_1 d + p_1(d\lambda_1)]e,$

$w_2' = [\lambda_2 d + p_2(d\lambda_2)]e.$

The Euler equation for equity is

$$p_s(d)u'(d) = \beta \left\{ \begin{matrix} (\phi_{s1} - v_s)[\lambda_1 d + p_1(\lambda_1 d)]u'(\lambda_1 d) + \\ (\phi_{s2} + v_s)[\lambda_2 d + p_2(\lambda_2 d)]u'(\lambda_2 d) \end{matrix} \right\}$$

The price of equity is still linear in d, and is now given by

$p_s(d) = k_s d$

where

$$k_s = \beta[(\phi_{s1} - v_s)\lambda_1^{1-\alpha}(1 + k_1) + (\phi_{s2} + v_s)\lambda_2^{1-\alpha}(1 + k_2)],$$

for $s = 1,2$.

Explicitly solving for k_1 and k_2, we obtain:

$$k_1 = \frac{\beta(\phi_{11} - a)\lambda_1^{1-\alpha}[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}] + \beta(\phi_{12} + a)\lambda_2^{1-\alpha} + \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}{[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}][1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] - \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}$$

$$k_2 = \frac{\beta(\phi_{22} + a)\lambda_2^{1-\alpha}[1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] + \beta(\phi_{21} - a)\lambda_1^{1-\alpha} + \beta^2(\phi_{21} - a)(\phi_{12} + a)(\lambda_1\lambda_2)^{1-\alpha}}{[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}][1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] - \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}$$

Thus, wealth in the next period is:

$w_1' = \lambda_1 d(1 + k_1),$

and

$w_2' = \lambda_2 d(1 + k_2).$

The price of the bond is given by

$$q_s(d) = \beta\left[\phi_{s1}\frac{1}{\lambda_1^\alpha} + \phi_{s2}\frac{1}{\lambda_2^\alpha} + a\left(\frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha}\right)\right]$$

for $s = 1,2$.

The conditional expected net return on equity is

$$ER_s^e = \frac{\phi_{s1}[\lambda_1 d + p_1(\lambda_1 d)] + \phi_{s2}[\lambda_2 d + p_2(\lambda_2 d)]}{p_s(d)} - 1$$

for $s = 1,2$, and the unconditional expected net return on equity ER^e, is

$\pi ER_1^e + (1 - \pi)ER_2^e - 1$

where the invariant probability π solves

$\pi = \phi_{11}\pi + \phi_{21}(1 - \pi).$

Therefore,

$$ER^e = \pi \frac{\phi_{11}[\lambda_1 d + p_1(\lambda_1 d)] + \phi_{12}[\lambda_2 d + p_2(\lambda_2 d)]}{p_1(d)} +$$
$$+(1-\pi)\frac{\phi_{21}[\lambda_1 d + p_1(\lambda_1 d)] + \phi_{22}[\lambda_2 d + p_2(\lambda_2 d)]}{p_2(d)} - 1$$
$$ER^e = \pi \frac{\phi_{11}\lambda_1(1+k_1) + \phi_{12}\lambda_2(1+k_2)}{k_1} +$$
$$(1-\pi)\frac{\phi_{21}\lambda_1(1+k_1) + \phi_{22}\lambda_2(1+k_2)}{k_2} - 1$$

The expected net return on the bond, R^b, is given by

$$\pi \frac{1}{q_1} + (1-\pi)\frac{1}{q_2} - 1 =$$
$$\frac{1}{\beta}\left[\frac{\pi}{\phi_{11}\frac{1}{\lambda_1^\alpha} + \phi_{12}\frac{1}{\lambda_2^\alpha} + a\left(\frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha}\right)} + \frac{(1-\pi)}{\phi_{21}\frac{1}{\lambda_1^\alpha} + \phi_{22}\frac{1}{\lambda_2^\alpha} + a\left(\frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha}\right)}\right] - 1.$$

Finally, the equity premium is given by

$$ER^e - R^b.$$